4/13/26

Exam next Wednesday. Covers through 5.3. (What we cover of it.)

Do part of § 5.3.

We've seen lots of lin transfs and studied the geometry of some.

Def A lin transf $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if it preserves the length of vectors, ie

$$\|T(\vec{x})\| = \|\vec{x}\| \quad \forall \vec{x} \in \mathbb{R}^n$$

We'll see other, equivalent, characterizations

The corresp matrix is an orthogonal matrix.

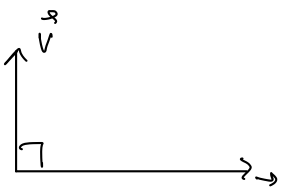
Ex $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ defines a rotation in $\mathbb{R}^2$ so the corresp T is orthogonal.

Th (5.3.2) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be an orthog linear transf. If $\vec{v}$ and $\vec{w}$ are orthog. Then so are $T(\vec{v})$ and $T(\vec{w})$.

so it not only preserves length, it also preserves orthogonality.

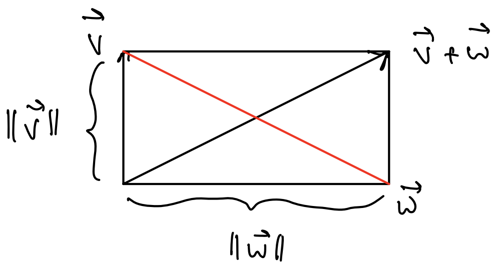
(Actually preserves angles)

pf/ We start with



(not necessarily the same length)

These define a rectangle. Note diagonals have the same length



$$\Rightarrow \boxed{\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v}+\vec{w}\|^2}$$ (Pythagoras). This holds because $\vec{v}, \vec{w}$ are orthogonal.

To show $T(\vec{v}), T(\vec{w})$ are orthogonal, we need to show

$$\boxed{\|T(\vec{v})\|^2 + \|T(\vec{w})\|^2 = \|T(\vec{v}) + T(\vec{w})\|^2} \quad (*)$$

Let's do that.

Since T is linear, $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$. So

$$\begin{aligned}
\|T(\vec{v})+T(\vec{w})\|^2 &= \|T(\vec{v}+\vec{w})\|^2 \\
&= \|\vec{v}+\vec{w}\|^2 \qquad \text{(T is orthog. so it preserves lengths.)} \\
&= \|\vec{v}\|^2 + \|\vec{w}\|^2 \qquad \text{(saw above)} \\
&= \|T(\vec{v})\|^2 + \|T(\vec{w})\|^2 \qquad \text{(again b/c T is orthog.)}
\end{aligned}$$

This is $(*)$, so we're done. //

thm (5.3.3)

(a) $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $T(\vec{e}_1), \dots, T(\vec{e}_n)$ form an orthonormal basis of $\mathbb{R}^n$

(b) An $n \times n$ matrix A is orthogonal iff its columns form an orthonormal basis of $\mathbb{R}^n$.

Note (b) follows from (a) and Th 2.1.2 : $A = \left[T(\vec{e}_1) \mid \cdots \mid T(\vec{e}_n) \right]$

I'll leave it to you to read pf of (a). //

Ex $A = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ A is orthogonal — check cols are orthonormal

So to check if a matrix is orthogonal, you don't have to check that it preserves lengths. Just check the columns! Sometimes shortens things.

Thm 5.3.4

If A and B are orthogonal then

- AB is orthogonal
- A^{-1} is orthogonal

pf/ Think about preserving length. Leave details to you. //

Def Let A be an $m \times n$ matrix.

The transpose of A, A^T, is the $n \times m$ matrix obtained by reversing the roles of rows + columns:

$$A = \begin{bmatrix} \text{——— row 1 ———} \\ \text{——— row 2 ———} \\ \vdots \\ \text{——— row m ———} \end{bmatrix}$$

$$A^T = \begin{bmatrix} | & | & & | \\ \text{row } 1 & \text{row } 2 & \cdots & \text{row } m \\ | & | & & | \end{bmatrix}$$

A square matrix is <u>symmetric</u> if $A = A^T$

<u>Ex</u> $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

<u>Ex</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

<u>Ex</u> $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is symmetric.

Now assume A is square. Note

dot prod $\qquad$ matrix mult.

$$\boxed{\vec{x} \cdot \vec{y} = \vec{x}^{\,T} \vec{y}}$$

Say $A = \begin{bmatrix} \vec{v}_1 & | & \vec{v}_2 & | & \cdots & | & \vec{v}_n \end{bmatrix}$

so

$$A^T = \begin{bmatrix} \vec{v}_1^{\,T} \\ \hline \vec{v}_2^{\,T} \\ \hline \vdots \\ \hline \vec{v}_n^{\,T} \end{bmatrix}$$

Then

$$A^T A = \begin{bmatrix} \vec{v}_1^{\,T} \\ \hline \vec{v}_2^{\,T} \\ \hline \vdots \\ \hline \vec{v}_n^{\,T} \end{bmatrix} \left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \cdots \,\middle|\, \vec{v}_n \right]$$

$$= \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vdots & \vdots & & \\ \vec{v}_n \cdot \vec{v}_1 & \vec{v}_n \cdot \vec{v}_2 & \cdots & \vec{v}_n \cdot \vec{v}_n \end{bmatrix}$$

<u>Thm</u> 5.3.7 (This is important.)

Let A be an $n \times n$ matrix. Then A is orthogonal iff $A^T A = I_n$,

ie iff $A^T = A^{-1}$.

pf/

A is orthogonal iff the columns $\vec{v}_i$ form an orthonormal basis (5.3.3)

$$\text{iff} \quad \vec{v}_i \cdot \vec{v}_i = 1 \quad \forall i$$
$$\vec{v}_i \cdot \vec{v}_j = 0 \quad \forall i \neq j$$

$$\text{iff} \quad A^T A = I_n \quad //$$

See Summary 5.3.8 about orthogonal matrices

→ key additional fact: orthogonal lin. transf. preserve dot products, hence angles, not only orthogonality. See below.

See Th 5.3.9 for a summary of basic facts about transposes